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LETTER TO THE EDITOR

Schur's Q-functions and the BKP hierarchy

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Abstract. In this letter we define Q-shift operators $\tilde{Q}_\lambda$ of Hirota derivatives that leads to a simple technique of constructing the hierarchy of BKP soliton equations, using Schur's Q-functions.

Solitons are nonlinear localized waves which

(i) propagate without changing their properties (shape, velocity, etc) and
(ii) are stable against mutual collisions in which each wave conserves its identity.

The solitary waves were first observed in a canal in August 1834, by a Scottish scientist and engineer named John Scott-Russell [2]. He proposed that the stability of the wave he had observed resulted from intrinsic properties of the wave's motion rather than from the circumstances of its generation. This view was not immediately accepted. In 1895 however, Korteweg and de Vries gave a complete analytic treatment of a nonlinear equation in hydrodynamics and showed that localized non-dissipative waves could exist. The name *soliton* was given because of its particle-like behaviour, although in elementary particle physics it is sometimes regarded as a field structure localized in space and time.

A modified form of the equation derived by Korteweg and de Vries, known as the KdV equation is

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

where the second term is a nonlinear term which acts to steepen the wave, whereas the third term is a dispersion term which spreads out the wave. As usual $u_y = \partial u/\partial y$. The balance between these opposing effects is the origin of the constant wave form and explains the existence of solitary waves.

There are a number of soliton equations but here we will study a particular type of soliton equation called the KP hierarchy of B-type (BKP for short) in Hirota form [1]. In partition notation the Hirota derivatives are restricted to odd part partitions only. Then

$$(D_{1^6} - 5D_{31^3} - 5D_{3^2} + 9D_{51})\tau \cdot \tau = 0 \tag{2}$$

where the D_λ, given by

$$D_x^m D_t^n \sigma \circ \tau \equiv \left(\frac{\delta}{\delta x} - \frac{\delta}{\delta x'}\right)^m \left(\frac{\delta}{\delta t} - \frac{\delta}{\delta t'}\right)^n \sigma(x,t)\tau(x',t')\bigg|_{x'=x,t'=t}$$

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are the Hirota derivatives.

It was shown in [1] that the polynomial solutions of the above equation are

$$\tau = L_\lambda(2x; -1)$$

for any partition λ into distinct parts, where the symmetric functions $L_\lambda(2x; -1)$ are directly related with the Schur Q-functions by a constant such that

$$L_\lambda(2x; -1) = 2^{|\lambda|} Q_\lambda(x; -1) .$$

From now on we will ignore the constant factor $2^{|\lambda|}$ for simplicity and will treat $L_\lambda(2x; -1)$ as a Schur Q-function and will denote it by Q_λ.

Before giving our main results we give some important definitions and for details we refer to [1, 8, 9].

Analogous to the *shift operators* defined in [9] we define the following.

Definition. The *Q-shift operator*, $\tilde{Q}_\lambda$ for a partition λ is defined as

$$\tilde{Q}_\lambda = \sum_\mu z_\mu^{-1} 2^{(l(\lambda)+l(\mu)+\epsilon)/2} \zeta_\mu^\lambda D_\mu$$

where the D_μ are Hirota derivatives.

From this definition, we may think of the Q-shift operator $\tilde{Q}_\lambda$ as analogous to the Schur's Q-functions and the Hirota derivatives D_μ as analogous to the power-sum symmetric functions.

It is easily verified that

$$\tilde{Q}_{(321)} = \tfrac{8}{45}(D_{(1^6)} - 5D_{(31^3)} + 9D_{(51)} - 5D_{(3^2)}) .$$

Hence, $\tilde{Q}_{(321)}$ gives rise to the first non-trivial solution of the BKP equation.

It is well known [1, 3, 9] that equations of odd degree are all trivial. Hence the BKP hierarchy will have the construction given in figure 1.

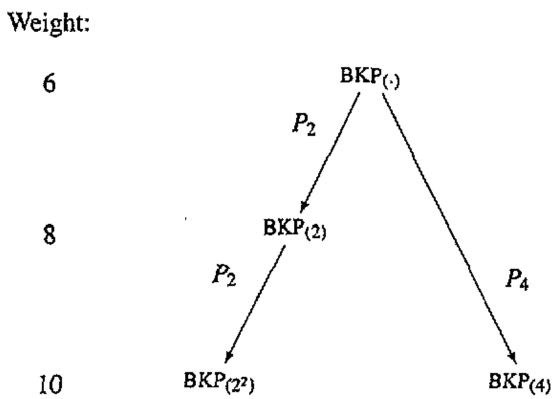


Figure 1.

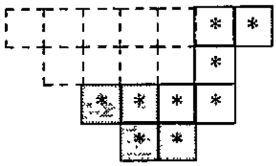
There will also be a new solution whenever the weight is an even triangular number (i.e. 6, 10, 28, 36, ...). These solutions are $\tilde{Q}_\lambda$ when $\lambda = (4321), (7654321), (87654321) \ldots$.

Definition. A skew diagram θ of a shifted diagram $D_{\lambda/\mu}$ is said to be a *strip* if it is connected and contains at most one box on every diagonal of the shifted diagram Y_λ. A skew diagram θ of a shifted diagram $D_{\lambda/\mu}$ is said to be a *double strip* if it is the union of two strips which both start on the first diagonal.

A box lying on the jth diagonal of $D^{\lambda/\mu}$ belongs to D_k if the intersection of $Y^{\lambda/\mu}$ and the jth diagonal has cardinality k. The *depth* of a double strip $Y^{\lambda/\mu}$ is defined to be

$$d(\lambda/\mu) = \frac{\mathrm{card}\{D_2\}}{2} + ht(D_1)\,.$$

Example. Suppose $\lambda = (7542)$ and $\mu = (54)$. Consider the double strip $D_{\lambda/\mu}$ below. The shaded boxes represent D_2.



The depth of $D_{\lambda/\mu}$ is 4.

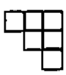
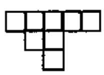
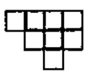
Theorem. From Józefiak [4] (theorem 2.4) it immediately follows that

$$D_r \tilde{Q}_\mu = \sum_\lambda 2^{l(\lambda)-l(\mu)} h^\lambda_{\mu(r)} \tilde{Q}_\lambda \tag{3}$$

where the summation is over all distinct part partitions λ such that $|\lambda| = |\mu| + r$ and $D_{\lambda/\mu}$ is a strip or double strip where

$$h^\lambda_{\mu(r)} = \begin{cases} (-1)^{ht(\lambda/\mu)} & \text{if} \quad D_{\lambda/\mu} \text{ is a strip} \\ 2(-1)^{d(\lambda/\mu)} & \text{if} \quad D_{\lambda/\mu} \text{ is a double strip}\,. \end{cases} \tag{4}$$

Example. Suppose $\tilde{Q}_\mu = (321)$. Then we have

$D_2[\tilde{Q}_{(321)}] = D_2$ () =  − 

We conjecture that (3) will give the BKP hierarchy for even values of r (see figure 2). We start with the solution of weight 6, $\tilde{Q}_{(321)}$.

For weight 8, we calculate the solution

$$\tilde{Q}_{(521)} - \tilde{Q}_{(431)} = \tfrac{8}{315}(D_{(1^8)} + 7D_{(31^5)} - 21D_{(51^3)} - 35D_{(3^21^2)} + 90D_{(71)} + -42D_{(53)})\,.$$

This solution is the same as reported by Jimbo [3].

Similarly, for weight 10, we calculate the solutions

$$\tilde{Q}_{(721)} + 2\tilde{Q}_{(541)} - \tilde{Q}_{(532)} - \tilde{Q}_{(631)} + 2\tilde{Q}_{(4321)} = \tfrac{16}{28350}(16S_1 - 30S_2 - 105S_3)$$

$$\tilde{Q}_{(721)} + \tilde{Q}_{(532)} - \tilde{Q}_{(631)} - 2\tilde{Q}_{(4321)} = \tfrac{16}{28350}(-10S_1 - 210S_2 + 105S_3)$$

$$\tilde{Q}_{(4321)} = \tfrac{16}{28350}(6S_1 - 90S_2)$$

where

$$S_1 = [D_{(1^{10})} + 63D_{(51^5)} - 225D_{(71^3)} - 175D_{(3^31)} + 525D_{(91)} - 189D_{(5^2)}]$$

$$S_2 = [6D_{(51^5)} - 5D_{(3^21^4)} - 15D_{(71^3)} + 15D_{(531^2)} - 5D_{(3^31)} + 10D_{(91)} - 15D_{(73)} + 9D_{(5^2)}]$$

$$S_3 = [D_{(31^7)} - 21D_{(531^2)} + 35D_{(91)} - 15D_{(73)}]\,.$$

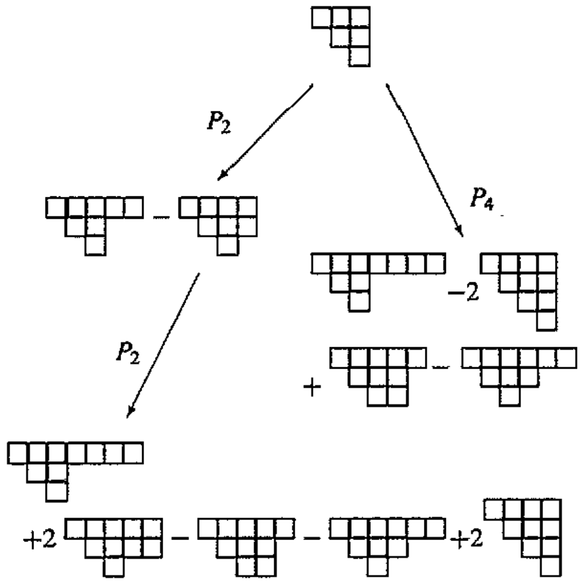


Figure 2. Construction of the BKP hierarchy.

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